Necessary and Sufficient Conditions for Consensusability of Linear Multi-Agent Systems

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***Abstract*—Consensusability of multi-agent systems (MASs) is a fundamental problem in the MAS research area, since when starting to design a consensus protocol, one should know whether or not there exists such a protocol that has the ability to make the MAS involved consensus. This technical note is aimed at studying the joint impact of the agent dynamic structure and the communication topology on consensusability. For the MASs with fixed topology and agents described by linear time-invariant systems, a necessary condition of consensusability with respect to a set of admissible consensus protocols is given, which is shown, under some mild conditions, to be necessary and sufficient.**

***Index Terms*—Consensusability, consensusability condition, consensus protocol, linear time-invariant (LTI) system, multi-agent system (MAS).**

I. Introduction

Recently, consensus problems for multi-agent systems (MASs) have been attracting lots of researchers, due to their broad applications in many areas such as swarms and flocks ([1]) and multi-vehicle systems ([2]), etc. In [3], a theoretical framework for consensus problems was provided for networked dynamic systems. In [4], consensus protocols were designed for both the first-order integral MASs and discrete-time MASs. Under some assumptions, the closed-loop systems were proved asymptotic consensus. In [5], the result was extended to the second-order case. It was shown that under some topology conditions, the protocols designed made the MAS asymptotic consensus in the context of fixed topologies and switching topologies, respectively. In [6], for linear MASs, output feedback consensus protocols were given, and the closed-loop MASs were shown to be asymptotic consensus if the topology had a spanning tree.

All the existing works ([7], [8]) on consensus problems focused on the design of consensus protocols and the closed-loop analysis. However, consensusability of MASs—a fundamental problem, which is concerned with the existence of consensus protocols, and of great importance in both synthesis and implementation of the protocols, is neither emphasized nor solved yet.

With respect to a given admissible control set, consensusability of MASs depends on two key factors, one is the dynamic structure of each agent, the other is communication topology among agents. When the dynamic structures of all the agents are fixed, one can investigate the relationship between the consensusability and the communication topology, such as in [3], [4], [9], [10], etc. Specifically, for different dynamic structures, to ensure the consensusability of the whole system, the communication topology conditions are different. Thus, it is worth analyzing the joint impact of the agent dynamic structure and the communication topology on consensusability. This technical note makes a first step towards this direction.

This technical note is aimed at consensusability of MASs. Linear time-invariant MASs (LTI-MASs) whose agents are described by LTI systems are a class of basic MASs. Hence, we take LTI-MASs as the breakthrough point to study consensusability. The communication topology within the LTI-MAS is fixed and represented by a digraph. Different from the undirected graph case, here the eigenvalues of the Laplacian matrix may be all complex since the Laplacian of a digraph is non-symmetric. Additionally, both the state matrix and input matrix of the agent dynamics in LTI-MAS are general matrices, which, generally speaking, cannot be transformed into Jordan canonical forms by one same linear transformation. These bring difficulties to the analysis of agent dynamic structure and communication topology conditions.

To overcome these difficulties, a linear transformation is introduced, which transfers the consensusability problem of LTI-MAS to the stability analysis of a closed-loop system. By using linear system theories, under some assumptions a necessary and sufficient condition on consensusability of LTI-MASs is obtained, that is, agents are stabilizable and detectable, and the topology has a spanning tree if the open-loop poles of each agent are not all in the open left half plane. This shows a relationship between consensusability of LTI-MASs and the dynamic and topology properties of the MASs.

The rest of this technical note is organized as follows. Some preliminary results of graph theory are briefly reviewed in Section II. The problem to be investigated is formulated in Section III. In Section IV, necessary and sufficient conditions on consensusability of LTI-MASs are given. In Section V, consensusability of discrete-time LTI-MASs is considered. In Section VI, some concluding remarks and open research topics are discussed.

The following notations will be used throughout this technical note. $\mathbb{R}^{m\times n}$ denotes the family of $m \times n$ dimensional real matrices. I_m denotes the $m \times m$ dimensional identity matrix. $\mathbb{R}$ denotes the real number field. $\mathbb{Z}^+$ denotes the nonnegative integer. $\mathbb{C}$ denotes the field of complex numbers. $\otimes$ denotes the Kronecker product. For a given vector or matrix X, X^T denotes its transpose; $\|X\|$ denotes its Euclidean norm; $\mathrm{Rank}(X)$ denotes its rank. For a square nonsingular matrix X, X^{-1} denotes its inverse matrix.

II. Preliminaries

Similar to [3], for convenience of description, we introduce the following terms. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ be a weighted digraph with the set of vertices $\mathcal{V} = \{1, 2, \cdots, N\}$ and the set of edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. In $\mathcal{G}$, the ith vertex represents the ith agent, and a directed edge from i to j is denoted as an ordered pair $(i, j) \in \mathcal{E}$, which means that agent j can directly receive information from agent i. In this case, the vertex i is called the parent vertex and the vertex j is called the child vertex. The set of neighbors of the ith agent is denoted by $N_i = \{j \in \mathcal{V} | (j, i) \in \mathcal{E}\}$. $\mathcal{A} = (a_{ij}) \in \mathbb{R}^{N\times N}$ is called the weighted adjacency matrix of $\mathcal{G}$ with nonnegative elements, and $a_{ii} = 0$, $a_{ij} > 0 \Leftrightarrow j \in N_i$. The in-degree of vertex i is defined as $deg_{in}(i) = \sum_{j=1}^{N} a_{ij}$ and the Laplacian of the weighted digraph $\mathcal{G}$ is defined as $L_{\mathcal{G}} = \mathcal{D} - \mathcal{A}$, where $\mathcal{D} = \mathrm{diag}(deg_{in}(1), \cdots, deg_{in}(N))$.

A directed tree is such a directed graph whose every vertex except the root, which has only children but no parent, has exactly one parent. A spanning tree of a digraph is a directed tree that contains all the vertices of the digraph.

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III. Consensusability and Admissible Protocols

Here we consider a system consisting of N agents indexed by $1, 2, \cdots, N$, respectively. The dynamics of the ith agent is described as follows:

$$\begin{aligned} \dot{x}_i(t) &= Ax_i(t) + Bu_i(t) \\ y_i(t) &= Cx_i(t), \quad i = 1, 2, \cdots, N \end{aligned} \tag{1}$$

where $x_i(t) \in \mathbb{R}^n$, $u_i(t) \in \mathbb{R}^r$ and $y_i(t) \in \mathbb{R}^m$ are the state, control and output of the ith agent, respectively; $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times r}$ and $C \in \mathbb{R}^{m\times n}$ are constant matrices.

With regarding the above N agents as vertices, the topology relationships among them can be conveniently described by a digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ with $\mathcal{V} = \{1, 2, \cdots, N\}$ and $\mathcal{A} = (a_{ij}) \in \mathbb{R}^{N\times N}$.

The consensus protocol for each agent in MASs is distributed ([3]) and only depends on the information of the agent itself and its neighbors, since each agent has limited capability of collecting information.

Roughly speaking, the consensusability of an MAS is concerned with the existence of a set of distributed consensus protocols such that all the states of the MAS asymptotically reach an agreement.

Nowadays, as described in [2]–[4], the following consensus protocol is often used:

$$u_i(t) = \sum_{j\in N_i} a_{ij}\left(x_j(t) - x_i(t)\right), \quad t \geq 0, \quad i = 1, 2, \cdots, N. \tag{2}$$

This protocol not only is distributed but also only depends on the errors of states between agent i and its neighbors. In fact, just as [11] pointed out, such errors are usually sufficient for consensus control. [12] considered the formation control of multi-vehicles, and achieved its control aim by using only the relative position information between agents and the virtual leader.

A characteristic feature of the consensus protocol (2) is to utilize all the relative information between agents' states and their neighbors'. However, in practice, due to economic costs or constraints on measurement, it is sometimes hard to directly measure the relative information of all the agents' states, but only the relative information of the agents' outputs is available. Hence, it would be more practical to consider the consensus protocol based on the outputs. Of course, when all the states are measurable, the consensus protocols based on the outputs and states are equivalent.

Thus, for simplicity, we will consider the output feedback case directly. Precisely, the consensus protocol of the ith agent is of the following form:

$$u_i(t) = K\sum_{j\in N_i} a_{ij}\left(y_j(t) - y_i(t)\right), \ t \geq 0, \ i = 1, 2, \cdots, N \tag{3}$$

where $K \in \mathbb{R}^{r\times m}$ is a weighted constant matrix. Noticing the property of a_{ij} that $a_{ij} > 0$ if and only if $j \in N_i$, (3) is equivalent to

$$u_i(t) = K\sum_{j=1}^{N} a_{ij}\left(y_j(t) - y_i(t)\right), \ t \geq 0, \ i = 1, 2, \cdots, N. \tag{4}$$

Let $u(t) = \left(u_1^T(t), u_2^T(t), \cdots, u_N^T(t)\right)^T$. We consider the following admissible control set:

$$\mathcal{U} = \left\{ u(t) : [0,\infty) \to \mathbb{R}^{rN} | u_i(t) = K\sum_{j=1}^{N} a_{ij}\left(y_j(t) - y_i(t)\right), \right. \\ \left. \forall t \geq 0, \quad K \in \mathbb{R}^{r\times m}, \quad i = 1, 2, \cdots, N \right\}. \tag{5}$$

Remark 1: When the consensus gain matrix $K = I_n$ and the output matrix of the system (1) is identical, i.e. $C = I_n$, the consensus protocol in (5) degenerates to the protocol (2) often used in the literature.

The admissible control set $\mathcal{U}$ covers a relatively large class of distributed consensus protocols. A natural question is that under what conditions, the MAS is consensusable with respect to (w.r.t.) such kind of admissible control set? To answer this question, we first give a definition of the consensusability of an MAS w.r.t. a given admissible control set $\mathcal{U}$.

Definition 1: For the system (1), if there exists a $u(t) \in \mathcal{U}$ such that for any initial value $x_i(0)$

$$\lim_{t\to\infty} \|x_j(t) - x_i(t)\| = 0, \quad i, j = 1, 2, \cdots, N$$

then we say that the system (1) is consensusable w.r.t. $\mathcal{U}$.

Remark 2: Different from the consensus definition in [13], where the states of all the agents are required to converge to one same constant value, here only the state differences between different agents are required to tend to zero, no matter whether the states themselves converge or not.

Remark 3: When all the eigenvalues of the state matrix A of the system (1) are in the open left half plane, the system (1) is naturally consensusable w.r.t. $\mathcal{U}$, since in this case, by simply taking $K = 0$ in (5) one can get that x_i $(i = 1, 2, \cdots, N)$ converges to zero exponentially, and hence, $\lim_{t\to\infty} \|x_j(t) - x_i(t)\| = 0, i, j = 1, 2, \cdots, N$. In view of this, unless otherwise stated, in this technical note, not all the eigenvalues of the state matrix A of the system (1) are in the open left half plane.

Next, we will demonstrate that the consensusability of LTI-MASs w.r.t. the admissible control set $\mathcal{U}$ depends on both the structure properties of each agent's dynamics and the topology within the MAS.

IV. Consensusability Conditions

Theorem 1: If the system (1) is consensusable w.r.t. $\mathcal{U}$, then (A, B, C) is stabilizable and detectable, and the topology $\mathcal{G}$ has a spanning tree.

Proof: By Definition 1, if the system (1) is consensusable w.r.t. $\mathcal{U}$, then there exist a matrix $K \in \mathbb{R}^{r\times m}$ and consensus protocols

$$u_i(t) = K\sum_{j=1}^{N} a_{ij}\left(y_j(t) - y_i(t)\right), \quad i = 1, 2, \cdots, N$$

such that for any $i \neq j$

$$\|x_j(t) - x_i(t)\| \to 0, \quad t \to \infty. \tag{6}$$

Let $\delta_i(t) \triangleq x_1(t) - x_i(t), i = 2, 3, \cdots, N$. Then, (6) is equivalent to $\|\delta_i(t)\| \to 0, t \to \infty, i = 2, 3, \cdots, N$.

Notice that

$$\dot{\delta}_i(t) = A\delta_i(t) + BKC\left[\sum_{j=1}^{N}(a_{ij} - a_{1j})\delta_j(t) - deg_{in}(i)\delta_i(t)\right], \\ i = 2, 3, \cdots, N.$$

Then, we have

$$\begin{aligned} \dot{\delta}(t) &\triangleq \begin{pmatrix} \dot{\delta}_2(t) \\ \vdots \\ \dot{\delta}_N(t) \end{pmatrix} \\ &= \left[I_{N-1} \otimes A - (L_{22} + \mathbf{1}_{N-1} \cdot \alpha^T) \otimes BKC\right]\delta(t) \end{aligned} \tag{7}$$

where $\mathbf{1}_{N-1}$ denotes an $N-1$ dimensional column vector with all components 1,

$$\alpha = (a_{12}, a_{13}, \cdots, a_{1N})^T,$$
$$L_{22} = \begin{pmatrix} deg_{in}(2) & -a_{23} & \cdots & -a_{2N} \\ -a_{32} & deg_{in}(3) & \cdots & -a_{3N} \\ \cdots & & \cdots & \\ -a_{N2} & -a_{N3} & \cdots & deg_{in}(N) \end{pmatrix}. \tag{8}$$

Thus, all the eigenvalues of $I_{N-1} \otimes A - (L_{22} + \mathbf{1}_{N-1} \cdot \alpha^T) \otimes BKC$ are in the open left half plane.

Take $S = \begin{pmatrix} 1 & 0 \\ \mathbf{1}_{N-1} & I_{N-1} \end{pmatrix}$. By the definition of Laplacian we have

$$S^{-1} L_{\mathcal{G}} S = \begin{pmatrix} 0 & -\alpha^T \\ 0 & L_{22} + \mathbf{1}_{N-1} \cdot \alpha^T \end{pmatrix}. \tag{9}$$

Assume that $\lambda_1 = 0, \lambda_2, \cdots, \lambda_N$ are the eigenvalues of Laplacian $L_{\mathcal{G}}$. Then, by (9), the eigenvalues of $L_{22} + \mathbf{1}_{N-1} \cdot \alpha^T$ are $\lambda_2, \cdots, \lambda_N$. Thus, there exists an invertible matrix T such that $L_{22} + \mathbf{1}_{N-1} \cdot \alpha^T$ is similar to a Jordan canonical matrix, i.e.

$$T^{-1}(L_{22} + \mathbf{1}_{N-1} \cdot \alpha^T)T = J = \mathrm{diag}(J_1, \cdots, J_s)$$

where J_k, $k = 1, 2, \cdots, s$, are upper triangular Jordan blocks, whose principal diagonal elements consist of λ_i, $i = 2, 3, \cdots, N$.

Therefore

$$(T \otimes I_n)^{-1} \left[I_{N-1} \otimes A - (L_{22} + \mathbf{1}_{N-1} \cdot \alpha^T) \otimes BKC \right] \times (T \otimes I_n) = I_{N-1} \otimes A - J \otimes BKC \tag{10}$$

is an upper triangular block matrix, which together with the properties of Kronecker product ([14]) implies that the eigenvalues of $I_{N-1} \otimes A - (L_{22} + \mathbf{1}_{N-1} \cdot \alpha^T) \otimes BKC$ are given by the eigenvalues of $A - \lambda_i BKC$, $i = 2, 3, \cdots, N$. Therefore, all the eigenvalues of $A - \lambda_i BKC$, $i = 2, 3, \cdots, N$ are in the open left half plane.

Now we prove (A, B, C) is stabilizable and detectable.

In fact, if at least one of λ_i, $i = 2, 3, \cdots, N$, is real, say λ_2, then (A, B, C) is stabilizable and detectable since all the eigenvalues of $A - \lambda_2 BKC$ are in the open left half plane. If all λ_i, $i = 2, 3, \cdots, N$, are complex numbers, that is, none of their imaginary parts are zeros, then noticing that $L_{22} + \mathbf{1}_{N-1} \cdot \alpha^T$ is a real matrix, the eigenvalues will appear in conjugate pair form. Without loss of generality, we assume that λ_2 and λ_3 are a pair of conjugate roots with $\lambda_2 = e + \jmath d$ and $\lambda_3 = e - \jmath d$ ($\jmath^2 = -1$). Noticing that $\forall \lambda \in \mathbb{C}$

$$\begin{vmatrix} \lambda I_n - (A - eBKC) & -dBKC \\ dBKC & \lambda I_n - (A - eBKC) \end{vmatrix} = |\lambda I_n - (A - \lambda_2 BKC)| \cdot |\lambda I_n - (A - \lambda_3 BKC)|$$

we can see that all the eigenvalues of $\begin{pmatrix} A - eBKC & dBKC \\ -dBKC & A - eBKC \end{pmatrix}$ are in the open left half plane, since all the eigenvalues of $A - \lambda_2 BKC$ and $A - \lambda_3 BKC$ are in the open left half plane. This together with

$$\begin{aligned} &\begin{pmatrix} A - eBKC & dBKC \\ -dBKC & A - eBKC \end{pmatrix} \\ &= \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} + \begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} -eKC & dKC \\ -dKC & -eKC \end{pmatrix} \\ &= \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} + \begin{pmatrix} -eBK & dBK \\ -dBK & -eBK \end{pmatrix} \begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix} \end{aligned}$$

implies that $\left(\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}, \begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix} \right)$ is stabilizable and $\left(\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}, \begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix} \right)$ is detectable. Thus

$$\mathrm{Rank} \begin{pmatrix} sI_n - A & 0 & B & 0 \\ 0 & sI_n - A & 0 & B \end{pmatrix} = 2n, \ \forall s \in \mathbb{C}, \ \mathrm{Re}s \geq 0 \tag{11}$$

where $\mathrm{Re}s$ denotes the real part of the complex number s. Combining this with

$$\mathrm{Rank} \begin{pmatrix} sI_n - A & 0 & B & 0 \\ 0 & sI_n - A & 0 & B \end{pmatrix} = 2\mathrm{Rank}(sI_n - A \quad B), \quad \forall s \in \mathbb{C}$$

gives

$$\mathrm{Rank}(sI_n - A \quad B) = n, \quad \forall s \in \mathbb{C}, \quad \mathrm{Re}s \geq 0.$$

Or equivalently, (A, B) is stabilizable.

Similarly, we can show the detectability of (A, C). Thus, (A, B, C) is stabilizable and detectable.

Now, we start to prove the second part of the theorem, that is, the topology $\mathcal{G}$ must have a spanning tree. In fact, from Lemma 3.3 of [4], we know that for all $i = 2, 3, \cdots, N$, $\lambda_i = 0$ or $\mathrm{Re}\lambda_i > 0$. Since the eigenvalues of A are not all in the open left half plane, then for all $i = 2, 3, \cdots, N$, $\lambda_i \neq 0$ must be true, since otherwise, there would be an $i \in \{2, 3, \cdots, N\}$ such that $\lambda_i = 0$, which in turn implies that all the eigenvalues of $A = A - \lambda_i BKC$ are in the open left half plane. This is a contradiction. Thus, by Lemma 3.3 of [4], $\mathcal{G}$ must have a spanning tree. ∎

Remark 4: To our knowledge, all the existing works ([4], [9], [10]) focused on the case where the agents are with specific dynamic structures, and some conditions in terms of fixed or time-varying communication topology are obtained for the consensus of the MASs considered. In contrast, here we aim at studying the relationship among the consensusability, the agent dynamic structure and the communication topology. From Theorem 1 one can see that, to ensure a consensus protocol, not only the topology of the MAS is required to have a spanning tree, but also the agents are required to be stabilizable and detectable. The former is a requirement on communication topology, while the latter is on the agent dynamic structure of the MAS.

From the proof of Theorem 1, it can be seen that by introducing a linear transformation, the consensusability of LTI-MASs can be converted into the stability problem of (7), which actually is equivalent to whether there exists a gain matrix $K \in \mathbb{R}^{r \times m}$ such that all the eigenvalues of $A - \lambda_i BKC$, $i = 2, 3, \cdots, N$ are in the open left half plane. The latter is essentially a static output feedback stability problem. Usually, only numerical solution to this problem can be given, and the analytical solution is involved with a Lyapunov inequality and an algebraic Riccati inequality ([15]).

Below we will show when the input/output matrix of the system (1) satisfies some rank criterion, the necessary condition in Theorem 1 is also sufficient.

As a matter of fact, if (A, B) is stabilizable, then from [16], the following Riccati equation

$$A^T P + PA - PBB^T P + I_n = 0 \tag{12}$$

has a unique nonnegative definite solution P, and furthermore, all the eigenvalues of $A - BB^T P$ are in the open left half plane.

Theorem 2: For the system (1), suppose that

$$\mathrm{Rank}(C) = \mathrm{Rank} \begin{pmatrix} C \\ B^T P \end{pmatrix} \tag{13}$$

where P is the nonnegative definite solution of (12). Then, the system (1) is consensusable w.r.t. $\mathcal{U}$ if and only if (A, B) is stabilizable, and the topology $\mathcal{G}$ has a spanning tree.

Proof: Necessity was shown in Theorem 1. Here we need only to prove the sufficiency.

Since the topology $\mathcal{G}$ has a spanning tree, from Lemma 3.3 of [4], we know that $\lambda_i, i = 2, 3, \cdots, N$ are all in the open right half plane. Namely, $\mathrm{Re}(\lambda_i) > 0, i = 2, 3, \cdots, N$. Denote

$$\delta \triangleq \min_{2 \le i \le N} \{\mathrm{Re}(\lambda_i)\} \tag{14}$$

where $\min\{S\}$ denotes the minimum one in S. By (13), the matrix equation

$$XC = B^T P$$

has solutions. Without loss of generality, we denote one of them by K_0. Take $K = \max\{1, \delta^{-1}\}K_0$ in the consensus protocol (4). Then, by (14) and

$$A - \lambda_i BKC = A - \lambda_i \max\{1, \delta^{-1}\} BB^T P, \quad i = 2, 3, \cdots, N$$

we know that all the eigenvalues of $A - \lambda_i BKC, i = 2, 3, \cdots, N$ are in the open left half plane since for any $\sigma \ge 1$ and $\omega \in \mathbb{R}$, all the eigenvalues of $A - (\sigma + \jmath\omega)BB^T P$ ($\jmath^2 = -1$) are in the open left half plane ([17]). This together with (7) and (10) implies that for all $i = 2, 3, \cdots, N$, $\|\delta_i(t)\| \to 0$, as $t \to 0$, or equivalently

$$\|x_j(t) - x_i(t)\| \to 0, \quad t \to 0, \quad i, j = 1, 2, \cdots, N.$$

Thus, by Definition 1, the system (1) is consensusable w.r.t. $\mathcal{U}$. ■

Remark 5: The sufficiency proof of Theorem 2 is constructive. By virtue of the solutions of the linear matrix equation $XC = B^T P$, the consensus protocols ensuring the consensus of the MAS are explicitly designed.

Remark 6: Apparently, if C is invertible, then (13) holds. It is worth pointing out that if (A, B) is stabilizable and (13) holds, then from the proof of Theorem 2, all the eigenvalues of $A - BK_0C$ are in the open left half plane. Thus, (A, C) is detectable.

Next, we use an example to demonstrate the application of the obtained results.

Example 1: Consider a system consisting of three agents in the plane, indexed by 1, 2, 3, respectively. The dynamics of the ith agent is described as follows:

$$\begin{aligned} \dot{x}_i(t) &= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} x_i(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u_i(t) \\ y_i(t) &= (0 \quad 1) x_i(t), \qquad i = 1, 2, 3 \end{aligned} \tag{15}$$

where $x_i(t) \in \mathbb{R}^2$, $u_i(t) \in \mathbb{R}$ and $y_i(t) \in \mathbb{R}$ are the state, control and output of the ith agent, respectively.

The topology among the above agents is described by the digraph $\mathcal{G}_1 = (\mathcal{V}_1, \mathcal{E}_1, \mathcal{A}_1)$, where $\mathcal{V}_1 = \{1, 2, 3\}$, $\mathcal{E}_1 = \{(2, 1), (2, 3), (3, 2)\}$ and $\mathcal{A}_1 = (a_{ij})_{3\times 3}$ is a nonnegative matrix with positive elements $a_{12} = a_{23} = 1$ and $a_{32} = 2$. Clearly, the topology $\mathcal{G}_1$ has a spanning tree.

It is easy to verify that the system (15) is stabilizable and condition (13) is satisfied. Hence, by Theorem 2, the system (15) is consensusable w.r.t. $\mathcal{U}$.

Next, we will choose protocols from $\mathcal{U}$ for the system (15) such that it achieves consensus. Take $K = 2$ in $\mathcal{U}$. Then, for any initial value, all the states of the three agents asymptotically reach an agreement. The simulation state trajectories are shown in Fig. 1, where the x-axis and y-axis form the plane and the axis perpendicular to the $x - y$ plane represents time. As time goes on, three agents asymptotically achieve consensus.

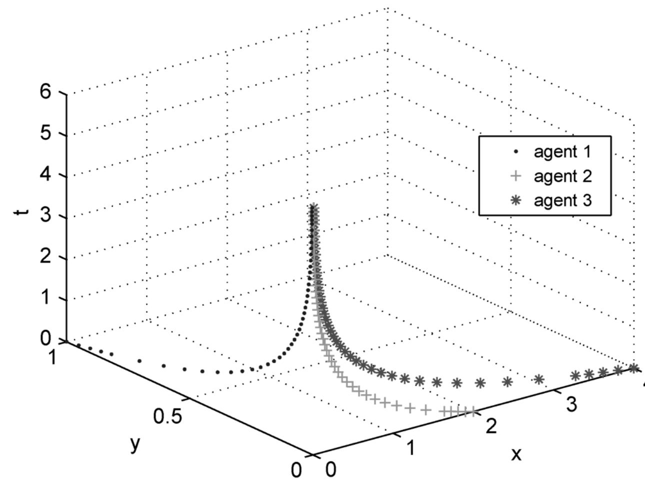

Fig. 1. State trajectories of three agents in the $x - y$ plane.

Specially, when $C = I_n$, condition (13) is naturally satisfied. Hence, by Theorem 2, we have the following corollary.

Corollary 1: The system (1) with $C = I_n$ is consensusable w.r.t. $\mathcal{U}$ if and only if (A, B) is stabilizable, and the topology $\mathcal{G}$ has a spanning tree.

Similar to Theorem 2, if (A, C) is detectable, then (A^T, C^T) is stabilizable. Thus, from [16], the following Riccati equation

$$A\overline{P} + \overline{P}A^T - \overline{P}C^T C\overline{P} + I_n = 0 \tag{16}$$

has a unique nonnegative definite solution $\overline{P}$.

Theorem 3: For the system (1), suppose that

$$\mathrm{Rank}(B) = \mathrm{Rank}(B \quad \overline{P}C^T) \tag{17}$$

where $\overline{P}$ is the nonnegative definite solution of (16). Then, the system (1) is consensusable w.r.t. $\mathcal{U}$ if and only if (A, C) is detectable, and the topology $\mathcal{G}$ has a spanning tree.

The proof is similar to that of Theorem 2, and hence, omitted here.

Remark 7: [6] studied the linear output feedback consensus problem of the system (1) with $B = I_n$. It was shown that if (A, C) was detectable and the topology $\mathcal{G}$ had a spanning tree, then a linear output feedback protocol like (4) existed and under which, the MAS considered was asymptotic consensus. Actually, by Theorem 3 we can further conclude that the sufficient condition in [6] is also necessary.

From the above analysis, one can see that the dynamic and communication properties of the agents are key to the consensusability of the LTI-MASs. Thus, with respect to a given admissible control set, the consensusability research for LTI-MASs, in essence, comes down to the studies on the dynamic and communication properties of the agents.

V. Consensusability of Discrete-Time Linear MASs

In the above sections, we studied the consensusability of the continuous-time LTI-MASs. Here we will consider the discrete-time case.

Consider the discrete-time LTI-MASs consisting of N agents. The dynamics of the ith agent is described as follows:

$$\begin{aligned} x_i(k+1) &= Gx_i(k) + Hu_i(k), \\ y_i(k) &= Fx_i(k),\ i = 1, 2, \cdots, N,\ k = 0, 1, \cdots \end{aligned} \tag{18}$$

where $x_i(k) \in \mathbb{R}^n$, $u_i(k) \in \mathbb{R}^r$ and $y_i(k) \in \mathbb{R}^m$ are the state, control and output of the ith agent at time k, respectively; $G \in \mathbb{R}^{n\times n}$, $H \in \mathbb{R}^{n\times r}$ and $F \in \mathbb{R}^{m\times n}$ are constant matrices.

Similar to the continuous-time case, we consider the following admissible control set:

$$\mathcal{U}^* = \Big\{ u(k) : \mathbb{Z}^+ \to \mathbb{R}^{rN} | u_i(k) = K \sum_{j=1}^{N} a_{ij} \left(y_j(k) - y_i(k) \right), \quad K \in \mathbb{R}^{r \times m}, \quad i = 1, 2, \cdots, N, \quad \forall k = 0, 1, \cdots \Big\}. \tag{19}$$

Remark 8: Different from the discrete-time consensus protocol in [4], here $u_i(k)$ only uses the relative output information between agent i and its neighbors, and moreover, $a_{ij} \geq 0$ $(i, j = 1, 2, \cdots, N)$ is defined as in Section III.

Many researchers ([3], [4], [10]) considered the discrete-time consensus problems for the MASs. For the first-order MASs of the following form:

$$x_i(t+1) = x_i(t) + u_i(t)$$

(and its continuous-time form) with fixed or time-varying topologies, [4] designed consensus protocols, and for the fixed topology case, proved that the system was asymptotic consensus if and only if the topology had a spanning tree. Recently, [10] considered the second-order systems of the following specific form:

$$\begin{pmatrix} \xi_i(t+1) \\ \zeta_i(t+1) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \xi_i(t) \\ \zeta_i(t) \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u_i(t)$$

with fixed or stochastic switching topologies. For the fixed topology case, it was proved that the MAS was asymptotically consentable with respect to a given set of admissible controls if and only if the network topology had a spanning tree.

The definition of consensusability for the system (18) is analogous to the continuous-time case.

Definition 2: For the system (18), if there exists a $u(k) \in \mathcal{U}^*$ such that for any initial value $x_i(0)$

$$\lim_{k \to \infty} \| x_j(k) - x_i(k) \| = 0, \quad i, j = 1, 2, \cdots, N$$

then we say that the system (18) is consensusable w.r.t. $\mathcal{U}^*$.

Theorem 4: When the system (18) is consensusable w.r.t. $\mathcal{U}^*$, if the eigenvalues of G are not all inside the unit circle, then the topology $\mathcal{G}$ must have a spanning tree, and furthermore, if G is nonsingular, then (G, H, F) is stabilizable and detectable.

Proof: By Definition 2, if the system (18) is consensusable w.r.t. $\mathcal{U}^*$, then there exists a matrix $K \in \mathbb{R}^{r \times m}$ and consensus protocols $u_i(k) = K \sum_{j=1}^{N} a_{ij}(y_j(k) - y_i(k)), i = 1, 2, \cdots, N, \forall k = 0, 1, \cdots$ such that for any $i \neq j$

$$\| x_j(k) - x_i(k) \| \to 0, \quad k \to \infty. \tag{20}$$

Denote $\delta_i(k) \triangleq x_1(k) - x_i(k), i = 2, 3, \cdots, N$, Then, (20) is equivalent to $\|\delta_i(k)\| \to 0, k \to \infty, i = 2, 3, \cdots, N$. Denote $\delta(k) \triangleq (\delta_2^T(k), \delta_3^T(k), \cdots, \delta_N^T(k))^T$. Then, we have

$$\delta(k+1) = \left[I_{N-1} \otimes G - (L_{22} + \mathbf{1}_{N-1} \cdot \alpha^T) \otimes HKF \right] \delta(k)$$

where L_{22} and α are defined as in (8). Thus, all the eigenvalues of $I_{N-1} \otimes G - (L_{22} + \mathbf{1}_{N-1} \cdot \alpha^T) \otimes HKF$ lie inside the unit circle. Noticing (10), we get all the eigenvalues of $G - \lambda_i HKF$, $i = 2, 3, \cdots, N$ lie inside the unit circle. If the eigenvalues of G are not all inside the unit circle, then the topology $\mathcal{G}$ must have a spanning tree. In fact, from Lemma 3.3 of [4], we know that for all $i = 2, 3, \cdots, N$, $\lambda_i = 0$ or $\mathrm{Re}\lambda_i > 0$. If the eigenvalues of G are not all inside the unit circle, then for all $i = 2, 3, \cdots, N$, $\lambda_i \neq 0$ must be true, since otherwise, there would be an $i \in \{2, 3, \cdots, N\}$ such that $\lambda_i = 0$, which in turn implies that all the eigenvalues of $G = G - \lambda_i HKF$ lie inside the unit circle. This is a contradiction. Thus, by Lemma 3.3 of [4], $\mathcal{G}$ must have a spanning tree.

Next, we only need to prove that (G, H, F) is stabilizable and detectable when G is nonsingular. In this case, similar to the proof of (11), from [16], one can get

$$\mathrm{Rank} \begin{pmatrix} sI_n - G & 0 & H & 0 \\ 0 & sI_n - G & 0 & H \end{pmatrix} = 2n, \ \forall s \in \mathbb{C}, \ |s| \geq 1 \tag{21}$$

or equivalently, $\mathrm{Rank}(sI_n - G \ H) = n, \forall s \in \mathbb{C}, |s| \geq 1$. Thus, (G, H) is stabilizable. Similarly, we can show the detectability of (G, F). Thus, (G, H, F) is stabilizable and detectable if G is nonsingular. ■

Remark 9: Different from the continuous-time case, for discrete-time LTI systems, when G is singular, (G, H) is stabilizable does not automatically imply $\mathrm{Rank}(sI - G \ H) = n, \forall s \in \mathbb{C}, |s| \geq 1$. This makes the proof method of (21) does not work for the case where G is singular.

Remark 10: In the continuous-time case, the sufficiency of consensusability can be proved by virtue of the solution of the Riccati equation (12) since for any $\sigma \geq 1$ and $\omega \in \mathbb{R}$, all the eigenvalues of $A - (\sigma + \jmath\omega)BB^T P$ $(\jmath^2 = -1)$ are in the open left half plane. Unfortunately, we fail to find the corresponding property in the discrete-time case.

VI. Conclusion

This technical note studies consensusability of LTI-MASs. In contrast to the existing works, the joint impact of the agent dynamic structure and the communication topology on consensusability is considered. By using the tools of the algebra, graph and linear system theory, some necessary and sufficient conditions on consensusability of LTI-MASs are provided.

It is worth noticing that this technical note is only a first step on consensusability study of MASs. Many important issues are still untouched and need to be investigated. For example, when the neighbor relationships among agents change over time (i.e. the topology is time-varying), when other kinds of admissible control sets are used, or when the agent dynamics of the MASs are different or nonlinear, what are the necessary and sufficient consensusability conditions.

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Functional Observability and the Design of Minimum Order Linear Functional Observers

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***Abstract*—The design of a minimum-order linear functional observer for linear time-invariant systems has been an open problem for over four decades. This technical note provides a solution to this problem. The technical note also introduces the concept of *Functional Observability/Detectability* and shows that the well-known concept of Observability/Detectability is a special case of *Functional Observability/Detectability*.**

***Index Terms*—Functional detectability, functional observability, functional observers, minimum order observers.**

I. Introduction

Linear functional observers estimate linear functions of the state vector of a system without estimating all the individual states. Such functional estimates are useful in feedback control system design because the control signal is often a linear combination of the states, and it is possible to utilize a linear functional observer to directly estimate the feedback control signal. The concept of linear functional observers has been around for more than four decades [1]. Yet to date, a method for designing a minimum-order linear functional observer has not been reported and this has remained an open problem [2]–[12] until now.

For a long time, it was understood that an observer that estimates linear functions of the state vector can have a lower order than that would be required to estimate the entire state vector. The concept of estimating linear functions was first explored by Luenberger [1]. Following the pioneering work of Luenberger [1], several other methods for solving single-functional and multi-functional observers were presented [2]–[16]. Darouach in [16] reported necessary and sufficient conditions for the existence and the design of an r-order linear functional observer where r is the number of linear functions to be estimated. Conditions to assure minimality of observers are also reported in [17], however those minimality conditions cannot always be satisfied, and hence a minimum observer structure cannot always be designed using previously reported results.

Despite all the contributions related to linear functional observers made so far, the problem of designing a minimum-order linear functional observer remained unsolved for over four decades. This technical note provides a solution to this problem. The proposed solution is based on recognizing: (i) Increasing the number of functions to be estimated increases the order of the observer by the same magnitude; and (ii) In order to estimate the desired linear functions, it is sometimes necessary to estimate extra linear functions as well as the desired linear functions. Consequently, the problem of designing a minimum-order linear functional observer is solved via finding the minimum number of extra linear functions that needs to be estimated. In solving the minimum-order observer problem, the technical note also introduces the

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